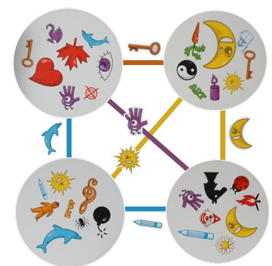
Make Spot It! with Finite Projective Planes

Science Mom and Math Dad

This handout accompanies Science Mom's The Math Behind **Spot It!** video. We cut out several of the math details that we wanted to include because the video was getting too long. But don't worry! We're including those details here.

The Game

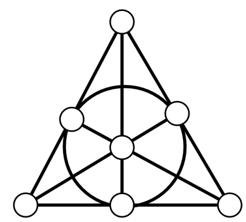
The game of **Spot It!** consists of 55 cards, each with 8 pictures per card. Players try to be the first to find a matching symbol on the cards to win the game. Amazingly, any two cards will always share exactly one image in common.



The game is fun, but the mathematical setup is even more interesting. How do we get cards that will always share one and only one image? It turns out that we need a mathematical structure called a finite projective plane.

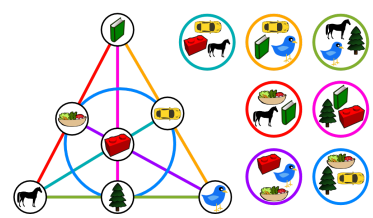
Building the Game

The simplest example of a finite projective plane is called the Fano plane. It consists of 7 points and 7 lines. (We count the circle as a line.) It satisfies the three properties of any projective plane.



1. Any two lines intersect at a unique point.
2. Any two points are connected by a line.
3. There are least four points, no three of which are on the same line.

We can use the Fano plane to build a 7-card game of **Spot It!** Assign images to each of the seven points. Then each of the seven lines determines a card.

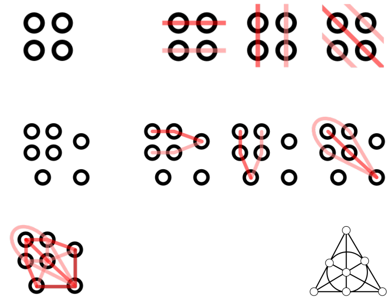


Property 1. guarantees that any two cards will share a unique image because any two lines of the Fano plane intersect at a unique point.

We can build a game from any finite projective plane in the exact same manner. Assign images to each point and then the lines will determine cards. We just have to understand how to build a finite projective plane.

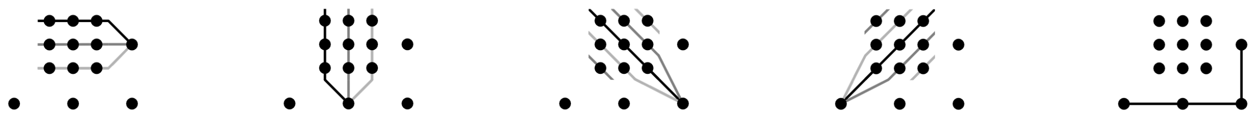
Constructing a Finite Projective Plane

The Fano plane can be constructed through a sequence of steps.



- Start with an $n \times n$ array of points. Then identify $n+1$ sets of n parallel lines that each pass through n of those points. (In this case "parallel" just means non-intersecting.)
- Add $n+1$ additional points and extend each set of parallel lines so that it meets one of the additional points.
- Connect each of the 3 additional points with a line. This gives us 7 points and 7 lines. We have built the Fano plane.

We can make another projective plane using $n = 3$. Start with a 3×3 array of points. Draw 4 sets of parallel lines (and let the parallel lines wrap around the figure). Add 4 additional points and extend each set of parallel lines to one of the new points. Also, connect the 4 new points.

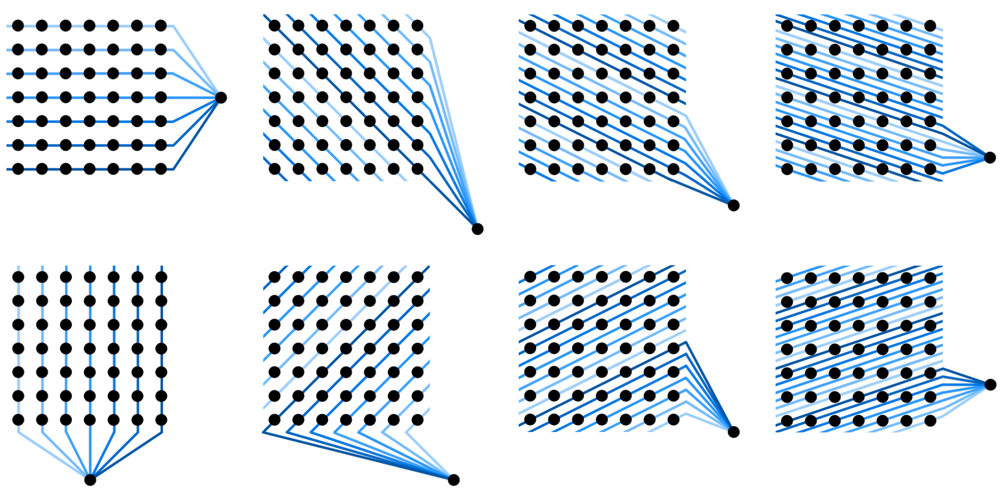


Those 13 lines and 13 points form a projective plane. We could build a game of **Spot It!** consisting of 13 cards and 13 images from this projective plane.

In general a finite projective plane will have $n^2 + n + 1$ points and $n^2 + n + 1$ lines.

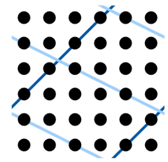
The Actual Game of ***Spot It!***

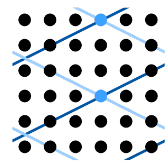
The actual game of **Spot It!** was created using the same construction with $n = 7$ (though the game **Spot It! Junior** uses $n = 5$).



Just assign 57 images to the 57 points. Then you get 57 cards determined by the 57 lines. In the actual game of **Spot It!** they have discarded two cards, so there are only 55 cards. We have no idea why they made the decision to toss out two cards.

Further Study





The construction presented here using diagonal lines only works when n is a prime number. If n is composite, the diagonal lines that are pointing in different directions might not intersect at all, or they might intersect at more than one point.

When n is a power of a prime number it is possible to create a finite projective plane using a finite field. The interested reader should look up the term *orthogonal latin squares* to learn more about how to get sets of "parallel lines" when n is a power of a prime.

Final Challenge

The colors of this 4×4 array define sets of "parallel lines." Use the construction we described (adding 5 additional points) to create a **Spot It!** game with 21 cards and 21 images.